

The Worst-Rank Frontier in Discrete Resource Allocation: No-Envy, Ordinal Efficiency, and Probabilistic Serial

Cemil Dibek ^{*} Furkan Doğan[†] Özgür Yılmaz[‡]

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Abstract

We study the smallest worst rank attainable in fractional allocation under no-envy and ordinal efficiency. The criterion bounds the rank of each object assigned with positive quantity. Pareto efficiency alone does not increase the minimum feasible rank. Under equal claims and arbitrary capacities, we characterize fixed-rank no-envy feasibility by an exact obstruction-and-recovery procedure and derive a direct rank-two system. Under unit claims and unit capacities, whenever no-envy is feasible at rank two, it is jointly feasible with ordinal efficiency at that rank. In this domain with n agents and n objects, the sharp worst-case rank costs of no-envy relative to feasibility and of efficiency conditional on no-envy are $\max\{0, n - 2\}$ and $\max\{0, n - 3\}$, respectively, while their instance wise sum never exceeds $\max\{0, n - 2\}$. For each $n \geq 3$, Probabilistic Serial can exceed each of the optimal ordinal-efficiency, no-envy, and joint frontiers by $n - 2$ ranks and by a factor of $n/2$; one family attains all six bounds. A heterogeneous-claim formulation based on feasible truncation explains why equal claims yield linear no-envy restrictions, whereas unequal claims can generate nonconvexity. All rank-loss bounds concern positive support, not quantity-weighted welfare.

Keywords: Discrete resource allocation; no-envy; worst rank; ordinal efficiency; Probabilistic Serial.

JEL classification: C78; D47; D63.

^{*}Koç University, Department of Industrial Engineering, İstanbul, 34450 Turkey. *E-mail address:* cdibek@ku.edu.tr.

[†]Koç University, Department of Economics, İstanbul, 34450 Turkey. Necmettin Erbakan University, Department of Economics, Konya, 42090 Turkey. *E-mail address:* fdogan19@ku.edu.tr.

[‡]Corresponding author. Koç University, Department of Economics, İstanbul, 34450 Turkey. *E-mail address:* ozyilmaz@ku.edu.tr. Personal webpage: mysite.ku.edu.tr/ozyilmaz

1 Introduction

How far down their preference rankings must agents go when an allocation is required to satisfy no-envy? We study this question through the social worst rank: the largest rank of an object assigned to an agent with positive quantity. A rank bound excludes all assignments below the corresponding cutoff, irrespective of their size. Our objective is to determine the smallest such bound compatible with no-envy and to separate the additional rank costs of efficiency and of selecting Probabilistic Serial.

We consider fractional allocations of capacitated objects to agents with ordinal preferences and positive claims. Feasibility requires each claim to be met and each object's capacity to be allocated. Preferences over allotments are defined by cumulative quantities at each preference threshold, equivalently by all strictly positive linear utility representations consistent with the ranking. We compare four frontiers: the minimum feasible worst rank, the minimum among Pareto-efficient allocations, the minimum under no-envy, and the minimum under both requirements. The general model permits heterogeneous claims and capacities. Equal claims and, subsequently, unit capacities yield additional structure.

We first establish the capacity benchmark. A weighted Hall condition characterizes feasibility within a given rank bound. Pareto efficiency does not raise this benchmark: there is a Pareto-efficient allocation attaining the minimum feasible rank. The reason is that an allocation dominating a rank-bounded allocation must itself meet each agent's full claim within the same rank bound. Maximizing a compatible linear welfare function within that bound therefore produces a globally Pareto-efficient allocation. Any subsequent increase in the frontier comes from no-envy or from imposing efficiency while retaining no-envy, not from efficiency alone.

Heterogeneous claims require a comparison convention. We compare an agent's allotment with the most preferred part of another agent's allotment that does not exceed the evaluating agent's claim. This feasible-truncation comparison supplies the fairness structure for the general model; it is not the main object of the paper. Its rank-prefix inequalities contain a saturation alternative, which can make the no-envy set nonconvex. Equal claims remove the alternative and reduce no-envy to direct linear comparisons of cumulative allotments. We use this distinction to identify the domain on which the subsequent fixed-rank analysis applies, without claiming that equal claims form the largest possible such domain.

Under equal claims and arbitrary capacities, we give an exact characterization of the no-envy frontier. At each rank, a transportation flow tests capacity feasibility. When capacity permits an allocation, fundamental-cycle coordinates eliminate the balance equalities. The remaining inequalities describe no-envy exactly. Elimination either produces a verifiable infeasibility certificate or permits the recovery of an allocation. Scanning ranks therefore determines

the no-envy frontier. This is a structural certificate-and-recovery characterization, not a new polynomial-time claim for Fourier–Motzkin elimination. Fixed-rank feasibility already has a polynomial-size linear formulation for rational data. At rank two, the conditions reduce further to a direct system describing displacement from first to second choices.

Unit claims and unit capacities turn the transportation problem into a matching problem. We characterize rank-two no-envy feasibility by a directed first–second-choice graph and construct an allocation whenever the conditions hold. Moreover, if no-envy can be attained using only the first two choices, ordinal efficiency can be imposed without increasing that rank. Unit capacities are essential to this conclusion: a three-agent example with unequal capacities has a rank-two no-envy allocation but no rank-two allocation satisfying both requirements.

This compatibility result identifies the threshold at which efficiency can become costly conditional on no-envy. Among strict unit-capacity instances with n agents and n objects, the sharp worst-case rank cost of no-envy relative to feasibility is $\max\{0, n - 2\}$, whereas the sharp cost of ordinal efficiency conditional on no-envy is $\max\{0, n - 3\}$. The two costs cannot accumulate arbitrarily: their sum in any instance is at most $\max\{0, n - 2\}$. Separate constructions attain the two individual bounds, and the combined bound is also sharp. These statements concern different maxima across instances; they do not assert that both individual maxima occur in the same instance.

We finally compare these frontiers with Probabilistic Serial. For each $n \geq 3$, one family has a no-envy and ordinally efficient allocation supported on the first two choices, while Probabilistic Serial assigns a positive share of a last choice. The resulting additive loss is $n - 2$ and the ratio is $n/2$ relative to each of the ordinal-efficiency, no-envy, and joint frontiers. All six bounds are sharp. The last-choice share in this family is $1/[2(n - 1)]$, so the comparison is deliberately one of absolute support. It does not imply a similar loss under a quantity-weighted objective or a criterion that ignores sufficiently small shares.

The general allocation model is motivated by the distribution of divisible resources from capacitated sources to recipients with heterogeneous claims and ordinal rankings over those sources. In this interpretation, a worst-rank bound limits the least-preferred source used to meet any recipient’s claim, while no-envy and efficiency discipline the comparison and reallocation of resource bundles. The analysis is stated in general allocation language and requires no application-specific cardinal utility or cost data. The application provides motivation and interpretation; the unit-claim, unit-capacity results describe a specialized assignment benchmark rather than the full heterogeneous environment.

1.1 Related Literature

Our paper relates to the literature on random assignment, in which agents rank objects and evaluate lotteries by stochastic dominance. In the unit-claim, unit-capacity domain, no-envy and ordinal efficiency are jointly attainable through Probabilistic Serial ([Bogomolnaia and Moulin, 2001](#)). Axiomatic characterizations provide further foundations for this mechanism ([Hashimoto, Hirata, Kesten, Kurino, and Ünver, 2014](#)). We retain these familiar allocation requirements and ask how they restrict the worst rank that can receive positive probability. The issue is therefore not whether fair and efficient assignments exist, but how far down agents’ rankings it is necessary to go to obtain them.

Several rank-based objectives evaluate assignments differently. Rank-maximal matching gives lexicographic priority to the numbers of agents receiving their first, second, and subsequent choices ([Irving, Kavitha, Mehlhorn, Michail, and Paluch, 2006](#)). Rank efficiency instead compares aggregate rank distributions by stochastic dominance ([Featherstone, 2020](#)), while rank-optimal assignment minimizes average rank, with particular attention to its behavior in uniform markets ([Nikzad, 2022](#)). Rank-sensitive refinements of ordinal efficiency also examine improvements in the probabilities of receiving higher-ranked objects ([Ramezani and Feizi, 2021](#)). Our criterion is a bound on positive support: no agent may receive any positive quantity of an object below the prescribed cutoff. It neither aggregates gains across agents nor discounts a low-ranked assignment because its share is small. Accordingly, our bounds concern the worst supported rank, not average rank or the entire rank distribution.

The concern for poorly assigned agents connects our analysis to Rawlsian and egalitarian approaches. In deterministic allocation, a Rawlsian criterion seeks to improve the assignment rank of the worst-off agents or reduce their number ([Afacan and Dur, 2024](#)). In random assignment, even-handedness offers a distinct egalitarian requirement, and its incompatibility with the joint imposition of stochastic-dominance efficiency and envy-freeness illustrates that different fairness criteria need not agree ([Duddy, 2025](#)). We do not identify no-envy with an egalitarian optimum. Instead, we treat no-envy as a constraint and minimize the social worst rank within the allocations satisfying it.

The closest comparison is with the Rawlsian approach to random assignment, which also establishes that stochastic-dominance envy-freeness is incompatible, in general, with minimizing the maximum rank assigned with positive probability ([Demeulemeester and Pereyra, 2026](#)). Thus, the qualitative conflict between no-envy and the minimum feasible worst rank is already known. Our contribution is to characterize the constrained frontier and quantify the associated rank costs. Under equal claims and arbitrary capacities, we give an exact fixed-rank obstruction-and-recovery characterization and a direct rank-two system. Under unit claims

and unit capacities, we establish rank-two compatibility between no-envy and ordinal efficiency and derive sharp bounds on the cost of no-envy, the additional cost of efficiency conditional on no-envy, and their sum. These results distinguish the separate effects of the two requirements rather than combining them into a single welfare comparison.

A further distinction concerns the selection of an assignment from the fair and efficient set. Conditions under which Probabilistic Serial is the unique ordinally efficient and envy-free assignment have been studied separately (Choi, 2016). Our question concerns its worst-rank performance when other assignments satisfying the same requirements are available. We compare its outcome with the minimum ranks attainable under ordinal efficiency, no-envy, and their conjunction. A single family attains the sharp additive and multiplicative losses against all three benchmarks. This is a loss from the mechanism's selection within the admissible set, not a failure of Probabilistic Serial to satisfy no-envy or ordinal efficiency.

Finally, heterogeneous claims connect the general model to multi-unit assignment. Generalizations of serial allocation accommodate agents who receive several objects (Kojima, 2009), and demand-normalized comparisons provide fairness requirements when demands differ across agents (Heo, 2014). Our comparison uses feasible truncation instead: an agent compares its allotment with the most preferred part of another agent's allotment that fits within its own claim. Under this convention, heterogeneous claims can generate nonconvex no-envy sets, whereas equal claims yield linear rank-prefix restrictions. This formulation supports the frontier analysis; it is not a characterization of a new serial rule. The sharp matching and Probabilistic Serial comparisons remain confined to the unit-claim, unit-capacity domain.

2 Model and Ordinal Preferences

Let $D = \{d_1, \dots, d_N\}$ and $S = \{s_1, \dots, s_M\}$ be disjoint finite sets of agents and objects, respectively, where N and M are positive integers. An object represents a type of resource whose supply can be allocated in divisible quantities. For each agent $d \in D$, let $q_d \in \mathbb{R}_{++}$ be its claim. For each object $s \in S$, let $q_s \in \mathbb{R}_+$ be its capacity. Total claims equal total capacity:

$$\sum_{d \in D} q_d = \sum_{s \in S} q_s. \quad (1)$$

For each object $s \in S$, an allocation function $\mu_s : D \rightarrow \mathbb{R}_+$ specifies the quantity of s assigned to each agent. An allocation is a profile $\mu = (\mu_s)_{s \in S}$ satisfying the following feasibility conditions:

$$\text{for each } s : \sum_{d \in D} \mu_s(d) = q_s, \quad (2)$$

$$\text{for each } d : \sum_{s \in S} \mu_s(d) = q_d. \quad (3)$$

The balance assumption allows feasibility to be written with equalities: each claim is met and all capacity is allocated. For each agent $d \in D$, its allotment is the vector $\mu(d) = (\mu_s(d))_{s \in S}$. A nonnegative vector $\mu' = (\mu'_s)_{s \in S}$ is weakly feasible for d if

$$\sum_{s \in S} \mu'_s \leq q_d.$$

Ordinal preferences over allotments. For each agent $d \in D$, let $\succeq_d$ be a preference ordering over S , with strict and indifference parts $\succ_d$ and $\sim_d$. Let $p(d)$ be the number of its indifference classes, denoted by $S_{d,1}, \dots, S_{d,p(d)}$ in decreasing order of preference. These classes are nonempty and partition S . For each $s \in S_{d,p}$ and $s' \in S_{d,p'}$, we have $s \succeq_d s'$ if and only if $p \leq p'$. For each subset $S' \subseteq S$, write $S'_{d,p} = S' \cap S_{d,p}$.

An allotment may contain quantities of several objects, so we extend each agent's ordering to allotment vectors. For each agent d and each class p , let u_d^p be the utility per unit of an object in $S_{d,p}$, and write $u_d = (u_d^1, \dots, u_d^{p(d)})$. We normalize the utility of receiving nothing to zero. Admissible utility-index vectors satisfy

$$u_d^1 > u_d^2 > \dots > u_d^{p(d)} > 0.$$

We then define the utility of an allotment for agent d as

$$U_d(\mu(d)) = \sum_{p \leq p(d)} u_d^p \times \sum_{s \in S_{d,p}} \mu_s(d). \quad (4)$$

This representation is additively separable and linear. We do not assume that the utility indices are observed. Instead, we compare allotments using all admissible utility-index vectors. For each agent d and each pair of weakly feasible allotments $\mu(d)$ and $\nu(d)$, write $\mu(d) R_d \nu(d)$ when all admissible vectors u_d satisfy

$$U_d(\mu(d)) = \sum_{p \leq p(d)} u_d^p \times \sum_{s \in S_{d,p}} \mu_s(d) \geq U_d(\nu(d)) = \sum_{p \leq p(d)} u_d^p \times \sum_{s \in S_{d,p}} \nu_s(d). \quad (5)$$

We write $\mu(d) P_d \nu(d)$ when the inequality is strict for each admissible utility-index vector.

We call R_d the ordinal preference relation over allotments. Strict positivity makes an additional unit valuable regardless of its type, while the differences between consecutive utility indices remain unrestricted. Moreover, R_d is generally an incomplete preorder on allotment vectors. It becomes a partial order only after allotments that differ solely in their distribution within indifference classes are identified. (For example, for $q_d = 50$, consider $\mu(d) = (24, 16, 10)$ and $\nu(d) = (20, 25, 5)$. If $u_d = (12, 9, 8)$, we have $U_d(\mu(d)) > U_d(\nu(d))$, whereas if $u_d = (10, 9, 8)$, we have $U_d(\mu(d)) < U_d(\nu(d))$.)

A cumulative-rank characterization of the preference relation follows directly from summation by parts. The positive utility gap between two consecutive preference ranks accounts for the corresponding rank-prefix comparison, while the strictly positive utility of the least-preferred indifference class accounts for the comparison of total quantities. The characterization therefore applies to weakly feasible allotments even when they contain different total quantities.

Proposition 1 *Let R_d be the preferences of d . Let $\mu(d)$ and $\nu(d)$ be two weakly feasible allotments for d . Then, $\mu(d) R_d \nu(d)$ if and only if for each $s' \in S$,*

$$\sum_{s: s \succ_d s'} \mu_s(d) \geq \sum_{s: s \succ_d s'} \nu_s(d). \quad (6)$$

3 Fairness, Efficiency, and Rank Frontiers

We adapt the no-envy criterion (Varian, 1974) to heterogeneous claims by comparing an agent's allotment with the most preferred part of another agent's allotment that is weakly feasible for the evaluating agent. The claim vector is exogenous, and the comparison does not normalize quantities by claims.

Fix an allocation μ and agents $d, d' \in D$. If $q_d < q_{d'}$, construct a feasible truncation of $\mu(d')$ for d by retaining quantities in decreasing order of d 's preference classes until their total reaches q_d . The retained vector is denoted by $\mu(d')|_d$. If the cutoff falls within an indifference class, the vector need not be unique, but all such truncations have the same class totals and are equivalent under R_d . If $q_d \geq q_{d'}$, set $\mu(d')|_d = \mu(d')$. Thus the comparison always uses the most preferred part of the competing allotment available within the evaluating agent's claim.

Definition 1 *An allocation μ satisfies no-envy if, for each $d, d' \in D$, we have $\mu(d) R_d \mu(d')|_d$.*

We next introduce the standard and most basic notion of efficiency.

Definition 2 *An allocation μ is Pareto efficient if there is no feasible allocation ν such that all agents $d \in D$ satisfy $\nu(d) R_d \mu(d)$ and at least one agent $d' \in D$ satisfies $\nu(d') P_{d'} \mu(d')$.*

The fairness comparison is conditional on the exogenous claim vector and therefore does not require equal total quantities across agents. Feasible truncation compares an agent with the most favorable part of another agent's allotment that is feasible under its own claim, making it the most demanding comparison among such feasible parts. A demand-normalized criterion would instead compare proportional claim satisfaction and would encode a different normative principle.

The preference and fairness definitions allow indifferences. Throughout the remainder of the paper, preferences over objects are strict. For each agent $d \in D$, object $s \in S$, and rank

$k \in \{1, \dots, M\}$, let $\text{rank}_d(s)$ be the position of s in d 's preference ordering and define $T_d(k) = \{s \in S : \text{rank}_d(s) \leq k\}$. For each feasible allocation μ , define the worst rank faced by agent d by $\rho_d(\mu) = \max\{\text{rank}_d(s) : \mu_s(d) > 0\}$, and the **social worst rank** by $\rho(\mu) = \max_{d \in D} \rho_d(\mu)$. Since each claim is positive, the set in the definition of $\rho_d(\mu)$ is nonempty.

The capacity frontier is the smallest social worst rank attained by a feasible allocation. The remaining frontiers impose Pareto efficiency, no-envy, or both:

1. $k^C = \min\{\rho(\mu) : \mu \text{ is feasible}\},$
2. $k^{PE} = \min\{\rho(\mu) : \mu \text{ is feasible and Pareto efficient}\},$
3. $k^{NE} = \min\{\rho(\mu) : \mu \text{ is feasible and satisfies no-envy}\},$
4. $k^{NE+PE} = \min\{\rho(\mu) : \mu \text{ is feasible, satisfies no-envy, and is Pareto efficient}\}.$

The minimum of an empty set is understood to be $+\infty$. The four quantities separate capacity feasibility, the rank cost of no-envy, and the additional cost of efficiency conditional on no-envy.

4 Capacity and Pareto Efficiency

We determine the capacity frontier in the general model and show that imposing Pareto efficiency does not change it.

For each $k \in \{1, \dots, |S|\}$, let $\mathcal{M}_k$ be the set of feasible allocations μ such that $\mu_s(d) = 0$ whenever $s \notin T_d(k)$. We call an allocation k -supported if it belongs to $\mathcal{M}_k$. For each $D' \subseteq D$, define $N_k(D') = \bigcup_{d \in D'} T_d(k)$.

Proposition 2 *For each $k \in \{1, \dots, |S|\}$, the set $\mathcal{M}_k$ is nonempty if and only if for each $D' \subseteq D$:*

$$\sum_{d \in D'} q_d \leq \sum_{s \in N_k(D')} q_s. \quad (7)$$

Proposition 2 is the capacitated analogue of Hall's theorem (Hall, 1935). It replaces cardinalities by total demand and total supply. It requires each collection of agents to have sufficient aggregate supply among the objects appearing within the first k positions of at least one agent in the collection. The next result shows that Pareto efficiency alone has no additional rank cost.

Theorem 1 *For each instance with strict preference orderings, $k^{PE} = k^C$.*

Theorem 1 shows that Pareto efficiency alone does not increase the capacity frontier. An additional rank cost can arise when no-envy is imposed, or when efficiency must be imposed while retaining no-envy.

Together with inclusion of the relevant feasible sets, Theorem 1 gives $k^C = k^{PE} \leq k^{NE} \leq k^{NE+PE}$. The two remaining gaps isolate the rank cost of no-envy and the additional cost of Pareto efficiency conditional on no-envy.

5 Claim Heterogeneity and No-Envy

We now examine how claim heterogeneity affects the structure of no-envy. Under heterogeneous claims, feasible truncation introduces an endogenous saturation condition into the rank-prefix comparisons, which can make the no-envy set nonconvex. Equal claims eliminate this saturation effect and reduce fixed-rank no-envy to direct linear rank-prefix inequalities.

For each $d, d' \in D$ and $s \in S$, let $F_d^{d'}(s; \mu) = \sum_{t: t \succeq_d s} \mu_t(d')$ denote the amount in agent d' 's allotment supplied by objects that agent d ranks weakly above s .

Lemma 1 *For each feasible allocation μ , each $d, d' \in D$, and each $s \in S$, the amount in the feasible truncation of d' 's allotment for d supplied by objects that d ranks weakly above s is $\min\{q_d, F_d^{d'}(s; \mu)\}$. Consequently, μ satisfies feasible-truncation no-envy if and only if, for each $d, d' \in D$ and $s \in S$,*

$$F_d^d(s; \mu) \geq \min \left\{ q_d, F_d^{d'}(s; \mu) \right\}. \quad (8)$$

The minimum in inequality (8) creates an endogenous saturation alternative. Since $F_d^d(s; \mu) \leq q_d$, inequality (8) is equivalent to

$$F_d^d(s; \mu) = q_d \quad \text{or} \quad F_d^d(s; \mu) \geq F_d^{d'}(s; \mu). \quad (9)$$

Indeed, if $F_d^{d'}(s; \mu) \leq q_d$, no-envy requires the second relation; if $F_d^{d'}(s; \mu) > q_d$, it requires the first. Consequently, for each fixed instance, the feasible-truncation no-envy set is a finite union of polyhedra corresponding to the alternative branches in (9).

The resulting union need not be convex. This failure arises from the fairness comparison itself and does not require a rank restriction or an efficiency condition.

Proposition 3 *The set of feasible allocations satisfying feasible-truncation no-envy need not be convex, even under strict preference orderings, unit object capacities, and rational agent claims.*

The proof in Appendix A provides two feasible-truncation no-envy allocations whose midpoint violates one of the prefix comparisons in Equation (8). The proposition shows that claim heterogeneity changes the structure of the fairness problem rather than merely its parameters. Equal claims eliminate the saturation alternative and restore direct rank-prefix comparisons.

Corollary 1 *Suppose that for each $d \in D$, $q_d = q > 0$. A feasible allocation μ satisfies no-envy if and only if, for each $d, d' \in D$ and $s \in S$,*

$$F_d^d(s; \mu) \geq F_d^{d'}(s; \mu). \quad (10)$$

Consequently, the no-envy set and each fixed-rank no-envy set are possibly empty polytopes. If μ is k -supported, only the inequalities associated with objects s satisfying $\text{rank}_d(s) < k$ need to be imposed.

Proof. Under equal claims, agent feasibility implies for each $d, d' \in D$ and $s \in S$, $F_d^{d'}(s; \mu) \leq q$. Hence, the minimum in Equation (8) equals $F_d^{d'}(s; \mu)$, which gives Equation (10). The resulting sets are described by finitely many linear equalities and inequalities and are bounded by feasibility. They are therefore polytopes.

Suppose that μ is k -supported. For each object s satisfying $\text{rank}_d(s) \geq k$, the prefix determined by s contains $T_d(k)$. Hence, $F_d^d(s; \mu) = q$. Since $F_d^{d'}(s; \mu) \leq q$, the corresponding no-envy inequality holds automatically. It therefore suffices to impose the inequalities for which $\text{rank}_d(s) < k$. ■

Equal claims therefore provide a domain-wide guarantee that no-envy is governed by direct linear rank-prefix comparisons. The next section uses this structure to derive fixed-rank obstructions and a constructive recovery procedure.

6 Equal Claims and Arbitrary Capacities

We now assume equal claims and normalize their common value to one. For each $d \in D$, we have $q_d = 1$. Object capacities remain arbitrary and satisfy

$$q_s \geq 0 \quad (s \in S), \quad \sum_{s \in S} q_s = |D|.$$

The normalization divides all claims, capacities, and allocation entries by the common claim. It preserves support ranks and no-envy comparisons. We first characterize no-envy feasibility at a fixed rank and then derive a direct characterization for rank two.

6.1 Fixed-Rank Obstruction and Recovery

The fixed-rank cycle system. For each $k \in \{1, \dots, |S|\}$, define the fixed-rank transportation graph, $G_k = (D \cup S, E_k)$ where $E_k = \{(d, s) \in D \times S : s \in T_d(k)\}$. Let $\mathcal{M}_k$ be the set of feasible allocations supported on E_k , and let $\mathcal{E}_k \subseteq \mathcal{M}_k$ be the subset satisfying no-envy. For each $A \subseteq D$, write $N_k(A) = \bigcup_{d \in A} T_d(k)$.

Proposition 2 specializes to this domain as follows: the set $\mathcal{M}_k$ is nonempty if and only if for each $A \subseteq D$, $|A| \leq \sum_{s \in N_k(A)} q_s$. When the condition holds, a maximum-flow computation returns a reference allocation $\mu^0 \in \mathcal{M}_k$. When it fails, the corresponding set A is a capacity obstruction at rank k . We first eliminate the agent- and object-balance equalities by parametrizing all balanced edge vectors through independent cycle coordinates. This reduction is useful since every remaining restriction is then an inequality in independent cycle coordinates.

Let c_k be the number of connected components of the full graph G_k , including isolated vertices. Choose a spanning forest F_k of G_k : within each connected component, F_k contains every vertex, connects the component, and contains no cycle. For each edge $e \in E_k \setminus F_k$, adding e to the forest creates one cycle. Since G_k is bipartite, this cycle has even length. Define its signed incidence vector $c^e \in \{-1, 0, 1\}^{E_k}$ by assigning $+1$ to e and then alternating $-1, +1$ around the cycle; all edges outside the cycle receive zero. Let $m_k = |E_k| - |D| - |S| + c_k$ and let C_k be the $|E_k| \times m_k$ matrix whose columns are these signed cycle vectors. If $m_k = 0$, C_k has no columns and no cycle coordinate is required.

Lemma 2 *Suppose that $\mathcal{M}_k \neq \emptyset$ and fix $\mu^0 \in \mathcal{M}_k$. An edge vector μ satisfies all agent and object equalities on G_k if and only if there is a unique $z \in \mathbb{R}^{m_k}$ such that*

$$\mu = \mu^0 + C_k z. \quad (11)$$

Appendix A proves the lemma from the fundamental-cycle basis of the graph. The forest is chosen from G_k itself, rather than from the positive support of μ^0 .

Extend each edge vector $y \in \mathbb{R}^{E_k}$ by setting $y_s(d) = 0$ whenever $(d, s) \notin E_k$. For each ordered comparison $a = (d, d', r)$ with $d \neq d'$ and $r \in \{1, \dots, k-1\}$, define the linear prefix deficit

$$\delta_a(y) = \sum_{s \in T_d(r)} (y_s(d') - y_s(d)). \quad (12)$$

For each $\mu \in \mathcal{M}_k$, Corollary 1 implies that μ satisfies no-envy if and only if all ordered comparisons a satisfy $\delta_a(\mu) \leq 0$. Substituting (11) gives the following cycle-coordinate System (13):

$$\begin{aligned} &\text{for each } e \in E_k : \quad (C_k z)_e \geq -\mu_e^0, \\ &\text{for each } e \in E_k : \quad (C_k z)_e \leq 1 - \mu_e^0, \\ &\text{for each ordered comparison } a : \quad \delta_a(\mu^0) + \delta_a(C_k z) \leq 0. \end{aligned} \quad (13)$$

The upper edge bounds are redundant since each agent receives one unit, but retaining them makes every cycle coordinate explicitly bounded. Indeed, the chord edge indexing a fundamental cycle belongs to no other fundamental cycle, so its lower and upper edge bounds give finite lower and upper bounds on the corresponding coordinate.

Write System (13) as $A_k z \leq b_k$ and let p_k be its number of rows. We now describe the complete elimination procedure for System (13), together with the associated infeasibility certificate and allocation-recovery operation.

Eliminating one coordinate. Suppose that the current system is written in the coordinate z_j and the vector w of remaining coordinates. Write each row i as $a_i^\top w + \alpha_i z_j \leq \beta_i$. Partition the rows according to the sign of the coefficient of z_j :

$$I_j^+ = \{i : \alpha_i > 0\}, \quad I_j^- = \{i : \alpha_i < 0\}, \quad I_j^0 = \{i : \alpha_i = 0\}.$$

Each row in I_j^0 is retained. For each $p \in I_j^+$ and $h \in I_j^-$, multiply row p by $-\alpha_h$, multiply row h by α_p , and add the resulting inequalities. This eliminates z_j and produces

$$[(-\alpha_h)a_p + \alpha_p a_h]^\top w \leq (-\alpha_h)\beta_p + \alpha_p \beta_h. \quad (14)$$

The retained rows together with all inequalities generated by (14) describe the exact projection of the current feasible set onto the coordinates in w . If either I_j^+ or I_j^- is empty, the rows in the nonempty one-sided class impose no restriction on this projection and are discarded.

Elimination and certificates. Let p_k be the number of rows in the original system $A_k z \leq b_k$. Assign the unit vector $e_i \in \mathbb{R}^{p_k}$ to its i th row. A retained row keeps its multiplier record. If rows p and h have multiplier records π_p and π_h , respectively, assign the generated row the record $\pi_{ph} = (-\alpha_h)\pi_p + \alpha_p \pi_h$.

Since $-\alpha_h > 0$ and $\alpha_p > 0$, each multiplier record remains nonnegative. Moreover, a row with record π is precisely the inequality obtained by taking the nonnegative linear combination $\pi^\top A_k z \leq \pi^\top b_k$.

Eliminate the coordinates in the following order:

$$z_{m_k}, z_{m_k-1}, \dots, z_1.$$

For each $j \in \{0, \dots, m_k\}$, let $\mathcal{P}_k^{(j)}$ denote the system in $(z_1, \dots, z_j)$ obtained after eliminating $z_{m_k}, \dots, z_{j+1}$. Thus, $\mathcal{P}_k^{(m_k)}$ is the original system and $\mathcal{P}_k^{(0)}$ consists only of constant inequalities. If $m_k = 0$, the original system already has this form. Suppose that $\mathcal{P}_k^{(0)}$ contains a false inequality $0 \leq \beta$ such that $\beta < 0$ and let λ be its multiplier record. Then,

$$\lambda \geq 0, \quad \lambda^\top A_k = 0, \quad \lambda^\top b_k = \beta < 0. \quad (15)$$

Hence, λ is a Farkas certificate for the cycle-coordinate system and proves that $\mathcal{E}_k = \emptyset$.

The certificate in (15) applies directly to the reduced system in cycle coordinates. Lemma 6 in Appendix A lifts the certificate in (15) to the original transportation formulation, in which the agent- and object-balance equalities require unrestricted multipliers.

Backward recovery. Suppose that $\mathcal{P}_k^{(0)}$ contains no false constant inequality. Restore the coordinates in the order $z_1, z_2, \dots, z_{m_k}$. At stage j , suppose that $z_1^*, \dots, z_{j-1}^*$ have already been selected. Each row i of $\mathcal{P}_k^{(j)}$ can be written as follows:

$$\sum_{\ell < j} a_{i\ell} z_\ell + a_{ij} z_j \leq \beta_i. \quad (16)$$

After substituting $z_1^*, \dots, z_{j-1}^*$, rows with $a_{ij} < 0$ provide lower bounds on z_j , while rows with $a_{ij} > 0$ provide upper bounds. Define

$$L_j = \max_{i: a_{ij} < 0} \frac{\beta_i - \sum_{\ell < j} a_{i\ell} z_\ell^*}{a_{ij}}, \quad (17)$$

$$U_j = \min_{i: a_{ij} > 0} \frac{\beta_i - \sum_{\ell < j} a_{i\ell} z_\ell^*}{a_{ij}}. \quad (18)$$

The chord-edge bounds retained in System (13) ensure that both collections are nonempty and that L_j and U_j are finite. Exactness of the preceding projection steps implies $L_j \leq U_j$. Set $z_j^* = L_j$ and proceed to the next coordinate.

After all coordinates have been restored, define $\mu^* = \mu^0 + C_k z^*$. The recovered vector z^* satisfies System (13). Hence, μ^* is nonnegative, satisfies all agent- and object-balance equalities, is supported on E_k , and satisfies no-envy. Therefore, $\mu^* \in \mathcal{E}_k$. If $m_k = 0$, no coordinates need to be restored. In that case, return μ^0 if every constant inequality in $\mathcal{P}_k^{(0)}$ holds.

The minimum-rank no-envy procedure. The fixed-rank no-envy sets are nested: $\mathcal{E}_1 \subseteq \mathcal{E}_2 \subseteq \dots \subseteq \mathcal{E}_M$. We determine the smallest nonempty set in this chain by scanning the rank levels in increasing order.

Minimum-Rank No-Envy Procedure

Step k Apply the maximum-flow test for the weighted Hall condition: for each $D' \subseteq D$,
 $|D'| \leq \sum_{s \in N_k(D')} q_s$.

Step $k.1$ If the maximum flow does not meet all agent claims, use a minimum cut to identify a set $D' \subseteq D$ satisfying $|D'| > \sum_{s \in N_k(D')} q_s$. Record this capacity obstruction, continue at Step $k + 1$.

Step $k.2$ If the weighted Hall condition holds, construct a reference allocation $\mu^0 \in \mathcal{M}_k$. Choose a spanning forest of G_k , form the fundamental-cycle matrix C_k , and construct System (13). Apply complete elimination while retaining the multiplier records.

- If elimination produces a false constant inequality, record the certificate in (15), continue at Step $k + 1$.

- If no false constant inequality appears, apply backward recovery, return the allocation, and terminate.

Let k^* denote the rank at which the procedure terminates.

Theorem 2 *For each $k \in \{1, \dots, |S|\}$, the fixed-rank procedure returns exactly one of the following:*

1. A set $D' \subseteq D$ satisfying $|D'| > \sum_{s \in N_k(D')} q_s$,
2. a multiplier record λ satisfying

$$\lambda \geq 0, \quad \lambda^\top A_k = 0, \quad \lambda^\top b_k < 0,$$

3. an allocation $\mu \in \mathcal{E}_k$.

The first two outputs occur only if $\mathcal{E}_k = \emptyset$, while the third output occurs if and only if $\mathcal{E}_k \neq \emptyset$. The rank scan terminates by rank $|S|$ and returns an allocation μ^* satisfying $\rho(\mu^*) = k^* = k^{NE}$. For each $k < k^*$, the procedure records an exact obstruction to rank- k no-envy feasibility.

Appendix A proves the theorem. The procedure is finite and exact under rational arithmetic. Literal Fourier–Motzkin elimination, however, need not run in polynomial time, because pairing positive and negative rows may generate exponentially many intermediate inequalities.

For rational input data, fixed-rank no-envy feasibility is a linear feasibility problem of polynomial input size and can therefore be solved in polynomial time by standard linear-programming methods. The role of elimination here is different: it makes the projected rank-prefix restrictions explicit, records how each infeasibility obstruction is assembled, and provides a direct recovery procedure for a feasible allocation.

6.2 A Direct Rank-Two Characterization

At rank two, the general cycle system collapses to a smaller formulation with a direct displacement interpretation. Assume that $|S| \geq 2$. For each agent d , let f_d and g_d be its first- and second-choice objects. For each $s, t \in S$, define $c_{st} = |\{d \in D : f_d = s, g_d = t\}|$ and $m_s = \sum_{t \in S} c_{st}$. Thus m_s is the number of agents ranking s first. Set $z_s = 0$ whenever $m_s = 0$.

Theorem 3 *Suppose that $|S| \geq 2$. The set $\mathcal{E}_2$ is nonempty if and only if there is a vector $z = (z_s)_{s \in S}$ satisfying the following:*

$$\text{for each } s \text{ with } m_s > 0 : \quad 0 \leq z_s \leq 1, \tag{19}$$

$$\text{for each } t \in S : \quad m_t(1 - z_t) + \sum_{s \in S} c_{st} z_s = q_t, \tag{20}$$

$$\text{whenever } c_{st} > 0 \text{ and } m_t > 0 : \quad z_s + z_t \leq 1. \tag{21}$$

Given such a vector, an allocation in $\mathcal{E}_2$ is recovered by setting, for each $d \in D$,

$$\mu_{f_d}(d) = 1 - z_{f_d}, \quad \mu_{g_d}(d) = z_{f_d}, \quad \mu_s(d) = 0 \quad \text{for } s \notin \{f_d, g_d\}. \quad (22)$$

If $|S| = 1$, total balance implies $q_s = |D|$ for the unique object s , and the unique feasible allocation $\mu_s(d) = 1$ satisfies no-envy.

The variables have a direct interpretation: z_s is the common amount displaced to the second choice of each agent that ranks s first. Equations (19)–(21) impose, respectively, variable bounds, object balance, and the no-envy restrictions. Arbitrary capacities change the efficiency conclusion at rank two. Unit capacities restore compatibility, as Section 7 shows, but equal claims alone do not.

Proposition 4 *There is a three-agent, three-object equal-claim instance satisfying*

$$k^{NE} = 2 < 3 = k^{NE+PE}. \quad (23)$$

Proposition 4 locates the role of the unit-capacity restriction. The restriction is not needed for the linear fixed-rank formulation or the rank-two z system. Unit capacities guarantee rank-two no-envy–efficiency compatibility; once they are removed, compatibility can fail.

7 Unit Capacities and Matching Structure

We now assume unit claims and unit capacities. Let $|D| = |S| = n$. For each $d \in D$, set $q_d = 1$, and for each $s \in S$, set $q_s = 1$. Feasible allocations are bistochastic and admit the standard random-assignment interpretation.

In this domain, we use the standard random-assignment terminology and refer to Pareto efficiency under stochastic dominance as ordinal efficiency. Accordingly, write $k^{\text{OE}} = k^{\text{PE}}$ and $k^{\text{NE+OE}} = k^{\text{NE+PE}}$.

The transportation structure of Section 6 now reduces to a matching structure. This additional structure yields a graph-based rank-two procedure and sharp bounds on the rank costs of no-envy relative to feasibility and of ordinal efficiency conditional on no-envy. Retain the fixed-rank graph G_k and the neighborhood notation $N_k(\cdot)$ introduced in Section 6.1.

Under unit object capacities, Proposition 2 reduces to the ordinary Hall condition: a k -supported feasible allocation exists if and only if $|D'| \leq |N_k(D')|$ for each $D' \subseteq D$. Whenever this condition holds, G_k contains a perfect matching, and the incidence vector of each perfect matching in G_k is a k -supported feasible allocation. The fixed-rank inequalities and the obstruction-and-recovery procedure developed in Section 6.1 remain valid. At rank two, however, no-envy feasibility can be decided without Fourier–Motzkin elimination. The resulting reduction depends only on the directed graph induced by first and second choices.

For later use, define the object-precedence relation $\triangleright_\mu$ as follows. For each $s, s' \in S$, write $s \triangleright_\mu s'$ if some agent receives a positive share of s' and strictly prefers s to s' . By Lemma 3, a feasible allocation is ordinally efficient if and only if $\triangleright_\mu$ is acyclic.

7.1 Rank-Two Characterization and Construction

Assume $n \geq 2$. For each agent $d \in D$, let a_d and b_d denote its first and second choices, respectively. Construct a directed multigraph H on S with one arc $a_d \rightarrow b_d$, labelled by d , for each agent d . For each $s, s' \in S$, define

$$e_{ss'} = |\{d \in D : a_d = s \text{ and } b_d = s'\}|, \quad m_s = \sum_{s' \in S} e_{ss'}.$$

Thus m_s is the number of agents whose first choice is s . Let $S^+ = \{s \in S : m_s > 0\}$ and $S^0 = S \setminus S^+$. For each $s \in S^+$ and $s' \in S$, define

$$P_{ss'} = \frac{e_{ss'}}{m_s}.$$

A nonempty set $C \subseteq S^+$ is a **closed class** if it is a strongly connected component of H and no arc leaves C . Let R be the union of all closed classes, and let $T = S^+ \setminus R$. Write P_T for the submatrix of P whose rows and columns are indexed by T , and write m_T for the restriction of m to T . Under Hall's condition for G_2 , each closed class is a directed cycle, and for each $s \in R$, $m_s = 1$. Moreover, $I - P_T^\top$ is nonsingular. Define the canonical displaced-flow vector $f \in \mathbb{R}^{S^+}$ by

$$f_s = 0 \quad \text{for each } s \in R, \quad f_T = (I - P_T^\top)^{-1} (m_T - \mathbf{1}). \quad (24)$$

For each $s \in R$, set $\beta_s = 0$ and for each $s \in T$, set

$$\beta_s = \frac{f_s}{m_s}.$$

The vector f is **admissible** if it satisfies the following four conditions:

$$\text{for each closed class } C \subseteq R : \quad \sum_{s' \in T} \sum_{s \in C} P_{s's} f_{s'} = 0, \quad (25)$$

$$\text{for each } s \in S^0 : \quad \sum_{s' \in T} P_{s's} f_{s'} = 1, \quad (26)$$

$$\text{for each } s \in T : \quad 0 \leq f_s \leq m_s, \quad (27)$$

$$\text{for each arc } s \rightarrow s' \text{ with } s, s' \in T : \quad \beta_s + \beta_{s'} \leq 1. \quad (28)$$

When Hall's condition holds and $T = \emptyset$, Hall's condition also implies $S^0 = \emptyset$. Equation (24) is then an empty system, and conditions (25)–(28) hold automatically. Given an admissible vector, define the canonical allocation $\hat{\mu}$ by

$$\hat{\mu}_{a_d}(d) = 1 - \beta_{a_d}, \quad \hat{\mu}_{b_d}(d) = \beta_{a_d}, \quad \hat{\mu}_s(d) = 0 \quad \text{for } s \notin \{a_d, b_d\}. \quad (29)$$

The rank-two displacement procedure.

Capacity test Construct G_2 and compute a maximum matching. If no perfect matching exists, use the alternating-path search from the unmatched agents to return a set $D' \subseteq D$ satisfying $|N_2(D')| < |D'|$, and terminate.

Displacement graph If a perfect matching exists, construct the directed multigraph H , the counts e and m , and the matrix P . Compute the strongly connected components of H , identify the closed classes, and form the sets R and T .

Propagation For each $s \in R$, set $f_s = \beta_s = 0$. Solve (24) for f_T , and for each $s \in T$, set $\beta_s = f_s/m_s$.

Structural tests Test conditions (25)–(28). If any condition fails, return the corresponding violation and terminate.

Allocation If all conditions hold, construct $\hat{\mu}$ according to (29), return $\hat{\mu}$, and terminate.

A failure in the capacity test is a Hall obstruction. A failure in the structural tests is a violated condition in the characterization below and therefore certifies that no rank-two no-envy allocation exists.

Theorem 4 *A rank-two no-envy allocation exists if and only if G_2 satisfies Hall's condition and the canonical displaced-flow vector satisfies conditions (25)–(28). Whenever these conditions hold, the allocation $\hat{\mu}$ defined in (29) is feasible, rank-two supported, and no-envy.*

The procedure requires one bipartite matching computation, one strongly connected-component decomposition, and one linear-system solve of order at most n . Standard implementations require $O(n^3)$ arithmetic operations. This is an arithmetic-operation bound and does not constitute a separate claim about bit complexity.

The vector $m - \mathbf{1}$ records excess first-choice demand. Equation (24) propagates this excess along second-choice arcs. An object that is nobody's first choice must absorb one unit, a closed first-choice cycle must receive no external displaced flow, and displacement into another first-choice object must satisfy the local no-envy bound. These are the only obstructions to rank-two no-envy feasibility.

7.2 Sharp Rank Costs of No-Envy and Efficiency

The unit-capacity domain permits sharp bounds on both the rank cost of no-envy relative to feasibility and the additional cost of ordinal efficiency conditional on no-envy. We first use the rank-two characterization to locate the point at which the latter cost can become positive.

Theorem 5 *For each strict unit-capacity instance, $k^{\text{NE}} \leq 2$ implies $k^{\text{NE+OE}} = k^{\text{NE}}$.*

The conclusion need not hold once unit object capacities are removed. The heterogeneous-capacity example in Proposition 4 satisfies $k^{\text{NE}} = 2 < 3 = k^{\text{NE+PE}}$.

Theorem 5 rules out an efficiency premium at ranks one and two. The following four-agent instance is the base case of the sharp bound once the no-envy frontier reaches rank three.

A four-agent base instance. Consider four agents d_1, d_2, d_3, d_4 and four objects s_1, s_2, s_3, s_4 , with preferences

$$\begin{aligned} d_1, d_2 : s_1 \succ s_2 \succ s_3 \succ s_4, \\ d_3 : s_1 \succ s_2 \succ s_4 \succ s_3, \\ d_4 : s_1 \succ s_4 \succ s_2 \succ s_3. \end{aligned} \tag{30}$$

At rank two, the set $\{d_1, d_2, d_3\}$ has only the objects s_1 and s_2 in its neighborhood, so Hall's condition fails. Hence, no rank-two feasible allocation exists. The unique rank-three no-envy allocation is

$$\mu^\dagger = \begin{pmatrix} 1/4 & 1/4 & 1/2 & 0 \\ 1/4 & 1/4 & 1/2 & 0 \\ 1/4 & 1/4 & 0 & 1/2 \\ 1/4 & 1/4 & 0 & 1/2 \end{pmatrix}, \tag{31}$$

where the rows correspond to d_1, d_2, d_3, d_4 , respectively, and the columns correspond to s_1, s_2, s_3, s_4 .

The allocation $\mu^\dagger$ is not ordinally efficient. In particular, d_3 receives a positive share of s_4 and prefers s_2 to s_4 , while d_4 receives a positive share of s_2 and prefers s_4 to s_2 . Exchanging equal positive shares of s_2 and s_4 between these two agents produces a stochastic-dominance improvement. The complete verification is provided in [Appendix A](#).

Theorem 6 *For each $n \geq 1$, among all strict unit-capacity instances with n agents and n objects,*

$$\begin{aligned} \max \{k^{\text{NE}} - k^C\} &= \max\{0, n - 2\}, \\ \max \{k^{\text{NE+OE}} - k^{\text{NE}}\} &= \max\{0, n - 3\}. \end{aligned} \tag{32}$$

Moreover, each such instance satisfies $k^{\text{NE+OE}} - k^C \leq \max\{0, n - 2\}$. The last bound is also sharp.

The first equality isolates the rank cost of no-envy relative to capacity feasibility, while the second isolates the additional cost of ordinal efficiency after no-envy. The final inequality shows that these losses cannot accumulate beyond $\max\{0, n - 2\}$. For each $n \geq 3$, the fairness and joint bounds are attained by a family with $(k^C, k^{\text{NE}}, k^{\text{NE+OE}}) = (2, n, n)$. For each $n \geq 4$, the conditional-efficiency bound is attained by the family with $(k^C, k^{\text{NE}}, k^{\text{NE+OE}}) = (3, 3, n)$ constructed in the proof.

8 Probabilistic Serial and Sharp Rank Losses

Throughout this section, claims and capacities are unitary and $|D| = |S| = n$. We compare Probabilistic Serial with the optimal no-envy and ordinal-efficiency frontiers in the same domain.

Probabilistic Serial. At time zero, each agent begins consuming at unit speed from its most-preferred object. Whenever an object is exhausted, each agent consuming from that object immediately moves to its most-preferred object with positive remaining capacity. If several objects are exhausted simultaneously, all corresponding moves occur simultaneously. Consumption continues until time one. For each $d \in D$ and $s \in S$, let $\mu_s^{PS}(d)$ be the length of time during which agent d consumes from object s . Since aggregate demand and aggregate capacity both equal n , the resulting matrix μ^{PS} is feasible.

Under strict preferences, Probabilistic Serial is envy-free and ordinally efficient (Bogomolnaia and Moulin, 2001). In the present unit-claim domain, envy-freeness is exactly no-envy as defined in Definition 1. Define $k^{PS} = \rho(\mu^{PS})$. Theorem 1, inclusion of the relevant feasible sets, and the properties of μ^{PS} imply

$$k^{OE} = k^C \leq k^{NE} \leq k^{NE+OE} \leq k^{PS}. \quad (33)$$

The successive gaps isolate the rank cost of no-envy, the additional cost of ordinal efficiency conditional on no-envy, and the rank cost of selecting Probabilistic Serial from the no-envy and ordinally efficient set. The final inequality can be strict even when the first three frontiers coincide. The following theorem determines the exact worst-case additive and multiplicative losses.

Theorem 7 *For each $n \geq 3$, the following statements hold.*

(i) *Over all strict unit-capacity instances with n agents and n objects,*

$$\max\{k^{PS} - k^{OE}\} = \max\{k^{PS} - k^{NE}\} = \max\{k^{PS} - k^{NE+OE}\} = n - 2. \quad (34)$$

(ii) *For each strict unit-capacity instance,*

$$\frac{k^{PS}}{k^{OE}} \leq \frac{n}{2}, \quad \frac{k^{PS}}{k^{NE}} \leq \frac{n}{2}, \quad \frac{k^{PS}}{k^{NE+OE}} \leq \frac{n}{2}. \quad (35)$$

All six bounds are sharp, and a single family of instances attains them simultaneously.

The tight family shows that the rank loss is not forced by no-envy or ordinal efficiency. An allocation satisfying both requirements uses only first and second choices, whereas PS assigns a positive amount to one agent from its last-ranked object. In that family, the last-choice share received by d_1 is $\mu_{s_n}^{\text{PS}}(d_1) = 1/[2(n-1)]$. This share is positive for each finite n but converges to zero as n grows. The bounds are therefore sharp for the paper's absolute-support criterion; they do not imply comparable losses under quantity-weighted performance measures or rank criteria that disregard sufficiently small shares.

9 Conclusion

We characterize how no-envy and efficiency affect the worst-rank frontier in fractional allocation. Pareto efficiency alone does not increase the minimum feasible rank. Under equal claims, no-envy has a linear fixed-rank formulation with exact obstruction and recovery; heterogeneous claims can destroy this convex structure. Unit capacities yield rank-two compatibility and sharp bounds on the separate and combined rank costs of no-envy and ordinal efficiency. Probabilistic Serial can attain its worst-case rank loss even when a no-envy and ordinally efficient allocation uses only the first two choices. These results distinguish the rank cost of fairness, the additional cost of efficiency conditional on fairness, and the cost of selecting a particular fair and efficient allocation. The bounds concern positive support and do not imply corresponding quantity-weighted welfare losses.

Appendix A Proofs

Proof of Proposition 1 Fix an agent $d \in D$. For each rank class $r \in \{1, \dots, p(d)\}$, define the difference in the amount assigned from that class by

$$x_r = \sum_{s \in S_{d,r}} (\mu_s(d) - \nu_s(d)).$$

Let $X_r = \sum_{\ell=1}^r x_\ell$ and set $u_d^{p(d)+1} = 0$. Since $x_r = X_r - X_{r-1}$, with $X_0 = 0$,

$$\sum_{r=1}^{p(d)} u_d^r x_r = \sum_{r=1}^{p(d)} u_d^r (X_r - X_{r-1}) = \sum_{r=1}^{p(d)} (u_d^r - u_d^{r+1}) X_r.$$

Summation by parts gives

$$U_d(\mu(d)) - U_d(\nu(d)) = \sum_{r=1}^{p(d)} (u_d^r - u_d^{r+1}) X_r. \quad (36)$$

For each $r \in \{1, \dots, p(d)\}$, set $c_r = u_d^r - u_d^{r+1}$. These coefficients are strictly positive, and the inverse transformation is

$$u_d^r = \sum_{\ell=r}^{p(d)} c_\ell.$$

Thus the admissible utility-index vectors correspond bijectively to $\mathbb{R}_{++}^{p(d)}$. If a prefix difference X_j is negative, choose $c_j = 1$ and, for each $r \neq j$, choose $c_r = \varepsilon > 0$. For sufficiently small ε ,

$$\sum_{r=1}^{p(d)} c_r X_r < 0.$$

Conversely, if all prefix differences are nonnegative, each admissible coefficient vector gives a nonnegative sum. The weak comparison therefore holds if and only if all prefix differences are nonnegative. The same argument shows that $\mu(d) P_d \nu(d)$ holds if and only if all prefix differences are nonnegative and at least one is positive. For each $r \in \{1, \dots, p(d)\}$ and $s' \in S_{d,r}$, the threshold s' determines the prefix $\bigcup_{\ell=1}^r S_{d,\ell}$. These are precisely the cumulative comparisons in the proposition. ■

Proof of Proposition 2 Fix k and write $Q = \sum_{d \in D} q_d$. Construct a network with a source, one node for each agent, one node for each object, and a sink. Give the arc from the source to agent d capacity q_d , give the arc from d to s capacity Q whenever $s \in T_d(k)$, and give the arc from object s to the sink capacity q_s . A flow of value Q saturates every source-to-agent arc. Since total object capacity also equals Q , it also saturates every object-to-sink arc. Hence, its agent-to-object flows define an allocation in $\mathcal{M}_k$. Conversely, every allocation in $\mathcal{M}_k$ induces a flow of value Q .

Let $D' \subseteq D$. The cut whose source side contains the source, the agents in D' , and the objects in $N_k(D')$ has capacity

$$\sum_{d \notin D'} q_d + \sum_{s \in N_k(D')} q_s.$$

If the weighted Hall condition fails for D' , this capacity is strictly below Q , so no flow of value Q exists.

Conversely, suppose that the weighted Hall condition holds. Consider any cut, and let D' be the agents on its source side. If an arc from an agent in D' to an object on the sink side crosses the cut, the cut has capacity at least Q . Otherwise, all objects in $N_k(D')$ lie on the source side, and the cut has capacity at least

$$\sum_{d \notin D'} q_d + \sum_{s \in N_k(D')} q_s \geq Q.$$

The max-flow–min-cut theorem therefore gives a flow of value Q . Such a flow saturates each source-to-agent arc. Since total supply also equals Q , it saturates each object-to-sink arc and defines a feasible allocation in $\mathcal{M}_k$. ■

Proof of Theorem 1 The inequality $k^C \leq k^{PE}$ follows because each Pareto-efficient allocation is feasible. Fix an allocation in $\mathcal{M}_{k^C}$. For each agent $d \in D$, choose strictly positive utility indices that strictly decrease along d 's preference ordering. Maximize the sum of the resulting linear utilities over the compact set $\mathcal{M}_{k^C}$, and let μ^* be a maximizer.

Suppose that a feasible allocation ν Pareto dominates μ^* . For each agent $d \in D$, the allocation $\mu^*(d)$ assigns its full claim within $T_d(k^C)$, thus

$$\sum_{s \in T_d(k^C)} \mu_s^*(d) = q_d.$$

By Proposition 1, cumulative-rank dominance implies

$$\sum_{s \in T_d(k^C)} \nu_s(d) \geq q_d.$$

Since the total allotment of d under ν is exactly q_d , equality must hold and for each $s \notin T_d(k^C)$, $\nu_s(d) = 0$. Hence $\nu \in \mathcal{M}_{k^C}$. The selected utility of each agent is weakly higher under ν than under μ^* and is strictly higher for at least one agent, contradicting the choice of μ^* . Hence μ^* is Pareto efficient and $k^{PE} \leq k^C$. ■

Proof of Lemma 1 Fix agents $d, d' \in D$ and an object $s \in S$. Suppose first that $q_d < q_{d'}$. The feasible truncation retains the best q_d units of $\mu(d')$ according to d 's ordering. If $F_d^{d'}(s; \mu) < q_d$, it retains each unit of $\mu(d')$ supplied by an object weakly preferred to s . If $F_d^{d'}(s; \mu) \geq q_d$, the truncation reaches its total size within that prefix. In either case, the truncated prefix equals $\min\{q_d, F_d^{d'}(s; \mu)\}$.

If $q_d \geq q_{d'}$, no truncation is required and $F_d^{d'}(s; \mu) \leq q_{d'} \leq q_d$, so the same formula holds. Proposition 1 identifies the preference relation with the cumulative comparisons at all object thresholds. Applying it to $\mu(d)$ and $\mu(d')|_d$ yields the stated characterization of no-envy. ■

Proof of Proposition 3 Let the agent claims be $(q_{d_1}, q_{d_2}, q_{d_3}, q_{d_4}) = (1, 2, 1, 1)$, let the five objects have unit capacity, and let the strict preferences be

$$\begin{aligned} d_1 &: s_1 \succ s_2 \succ s_3 \succ s_4 \succ s_5, \\ d_2 &: s_1 \succ s_4 \succ s_3 \succ s_5 \succ s_2, \\ d_3 &: s_2 \succ s_4 \succ s_5 \succ s_1 \succ s_3, \\ d_4 &: s_3 \succ s_4 \succ s_5 \succ s_1 \succ s_2. \end{aligned}$$

With rows ordered by d_1, d_2, d_3, d_4 and columns ordered by $s_1, \dots, s_5$, define

$$\mu^A = \begin{pmatrix} 1/2 & 1/4 & 0 & 1/4 & 0 \\ 1/2 & 0 & 1/4 & 1/4 & 1 \\ 0 & 3/4 & 0 & 1/4 & 0 \\ 0 & 0 & 3/4 & 1/4 & 0 \end{pmatrix}, \quad \mu^B = \begin{pmatrix} 1/2 & 1/3 & 1/6 & 0 & 0 \\ 1/2 & 1/3 & 5/12 & 1/3 & 5/12 \\ 0 & 1/3 & 0 & 1/3 & 1/3 \\ 0 & 0 & 5/12 & 1/3 & 1/4 \end{pmatrix}. \quad (37)$$

Each matrix has row sums $(1, 2, 1, 1)$ and unit column sums, so both are feasible.

For $X \in \{A, B\}$ and agents $d, d' \in D$, let $\mathbf{f}_d^X(d')$ list the five values $\min\{q_d, F_d^{d'}(s; \mu^X)\}$ as s runs from the first to the fifth object in d 's preference ordering. The following table gives, for each evaluator, its own vector and the componentwise maximum of the other three vectors:

X	d	$\mathbf{f}_d^X(d)$	$\max_{d' \neq d} \mathbf{f}_d^X(d')$
A	d_1	$(1/2, 3/4, 3/4, 1, 1)$	$(1/2, 3/4, 3/4, 1, 1)$
A	d_2	$(1/2, 3/4, 1, 2, 2)$	$(1/2, 3/4, 1, 1, 1)$
A	d_3	$(3/4, 1, 1, 1, 1)$	$(1/4, 1/2, 1, 1, 1)$
A	d_4	$(3/4, 1, 1, 1, 1)$	$(1/4, 1/2, 1, 1, 1)$
B	d_1	$(1/2, 5/6, 1, 1, 1)$	$(1/2, 5/6, 1, 1, 1)$
B	d_2	$(1/2, 5/6, 5/4, 5/3, 2)$	$(1/2, 1/2, 3/4, 1, 1)$
B	d_3	$(1/3, 2/3, 1, 1, 1)$	$(1/3, 2/3, 1, 1, 1)$
B	d_4	$(5/12, 3/4, 1, 1, 1)$	$(5/12, 3/4, 1, 1, 1)$

In each row, the own vector componentwise dominates the maximum competing vector. Lemma 1 therefore implies that μ^A and μ^B satisfy no-envy.

Let $\bar{\mu} = (\mu^A + \mu^B)/2$. At the third-rank threshold of d_1 , its own allotment has cumulative mass

$$F_{d_1}^{d_1}(s_3; \bar{\mu}) = \frac{1}{2} \left(\frac{3}{4} + 1 \right) = \frac{7}{8},$$

whereas the allotment of d_2 has cumulative mass $F_{d_1}^{d_2}(s_3; \bar{\mu}) = \frac{1}{2}(\frac{3}{4} + \frac{5}{4}) = 1$. Since $q_{d_1} = 1$, the no-envy inequality requires $7/8 \geq \min\{1, 1\} = 1$, a contradiction. Thus the no-envy set is not convex. ■

Proof of Lemma 2 Orient each edge of G_k from its agent endpoint to its object endpoint, and let B_k be the signed node–edge incidence matrix. If μ and μ^0 satisfy the same agent and object equalities, their difference $y = \mu - \mu^0$ satisfies $B_k y = 0$.

For each chord $e \in E_k \setminus F_k$, the fundamental-cycle vector c^e belongs to $\ker B_k$. Its coordinate on e equals one, while each other fundamental-cycle vector is zero on that chord. These vectors are therefore linearly independent. Their number is

$$|E_k| - (|D| + |S| - c_k) = m_k,$$

which is the dimension of $\ker B_k$. Hence the fundamental-cycle vectors form a basis of $\ker B_k$, and $y = C_k z$ for a unique $z \in \mathbb{R}^{m_k}$. Conversely, each column of C_k has zero incidence at each vertex, so $\mu^0 + C_k z$ preserves all agent and object equalities. ■

Lemma 3 (Trading-cycle criterion) *Fix a balanced instance with strict preference orderings and a feasible allocation μ . For objects $s, s' \in S$, write $s \triangleright_\mu s'$ if some agent receives a positive amount from s' and strictly prefers s to s' . The allocation μ is Pareto efficient if and only if $\triangleright_\mu$ is acyclic.*

Proof. Suppose that $s_1 \triangleright_\mu s_2 \triangleright_\mu \cdots \triangleright_\mu s_\ell \triangleright_\mu s_1$. For each arc, choose an agent that receives a positive amount from the object at the head of the arc and strictly prefers the object at its tail. Move a sufficiently small common amount around the cycle. Each agent and object total is preserved, and each participating agent replaces a quantity of one object by the same quantity of a strictly preferred object. The resulting allocation Pareto dominates μ .

Conversely, suppose that a feasible allocation ν Pareto dominates μ . For each agent $d \in D$, the cumulative-rank inequalities allow $\nu(d) - \mu(d)$ to be decomposed into nonnegative transfers from objects at which $\mu(d)$ is reduced to strictly preferred objects at which it is increased. Order the objects as $s_1 \succ_d \cdots \succ_d s_M$ and set $\Delta_r = \nu_{s_r}(d) - \mu_{s_r}(d)$. The prefix inequalities give $\sum_{\ell=1}^r \Delta_\ell \geq 0$ for each r , and feasibility gives $\sum_{\ell=1}^M \Delta_\ell = 0$.

Scan this ordering from best to worst, retaining the unmatched positive increments. Whenever $\Delta_r < 0$, match its magnitude with previously unmatched positive increments. Immediately before this step their total is $\sum_{\ell < r} \Delta_\ell \geq -\Delta_r$, so the matching is possible. Each match represents a transfer from s_r to a strictly preferred object. The zero total difference ensures that no unmatched increment remains at the end.

Represent each transfer by an arc from the preferred object to the relinquished object. Each such arc belongs to $\triangleright_\mu$ because the relinquished share is positive under μ .

At each object, feasibility of μ and ν makes the total weight of incoming transfer arcs equal the total weight of outgoing transfer arcs. Strict dominance for at least one agent makes this

balanced directed graph nonempty. A finite nonempty balanced directed graph contains a directed cycle, so $\triangleright_\mu$ is cyclic. ■

Lemma 4 (Exactness of the cycle system) *Suppose that $\mathcal{M}_k \neq \emptyset$. The map $z \mapsto \mu^0 + C_k z$ is a bijection from the solution set of the cycle-coordinate system onto $\mathcal{E}_k$.*

Proof. Lemma 2 gives a bijection between cycle coordinates and edge vectors satisfying all agent and object equalities. The two edge-bound rows of the cycle system are equivalent to $0 \leq \mu_e \leq 1$ for each $e \in E_k$. Edges outside E_k receive zero, so these rows impose nonnegativity and rank- k support. The remaining rows are exactly the rank-prefix no-envy inequalities from Corollary 1. The image of the solution set is therefore precisely $\mathcal{E}_k$. ■

Lemma 5 (Exact elimination and backward recovery) *Complete elimination produces a false constant inequality if and only if the cycle-coordinate system is infeasible. If no false constant inequality is produced, backward recovery returns a solution of the original system.*

Proof. Consider one elimination step, and write a current row as $a_r^\top w + \alpha_r z_j \leq \beta_r$, where w contains the coordinates not being eliminated. For each fixed w satisfying all rows with $\alpha_r = 0$, there exists a value of z_j satisfying all current rows if and only if each lower bound generated by a row with $\alpha_r < 0$ does not exceed each upper bound generated by a row with $\alpha_r > 0$. Pairing one row of each sign produces exactly these lower–upper comparisons. Rows with $\alpha_r = 0$ remain restrictions on w . If only one sign occurs, the corresponding one-sided bounds can be met by choosing z_j sufficiently far in the appropriate direction. The generated system is therefore the exact projection of the current feasible set onto w .

Applying this argument successively to $z_{m_k}, \dots, z_1$ shows that the final constant system is feasible if and only if the original cycle-coordinate system is feasible. Since the final system has no variables, it is feasible if and only if it contains no inequality $0 \leq \beta$ with $\beta < 0$.

Suppose that no contradiction appears. At recovery stage j , the selected values $z_1^*, \dots, z_{j-1}^*$ satisfy the projected system in these coordinates. Each lower–upper comparison generated when z_j was eliminated therefore holds, so $L_j \leq U_j$. The chord indexing the j th fundamental cycle appears in no other cycle vector. Its two edge-bound rows therefore involve only z_j and survive until this recovery stage, making L_j and U_j finite. Choosing $z_j^* = L_j$ satisfies each row that contains z_j , while each row independent of z_j already belongs to the preceding projected system. Induction over $j = 1, \dots, m_k$ recovers a vector satisfying the original system.

Finally, assign the unit record to each original row and update records by the nonnegative row combinations used in elimination. Induction over the elimination steps shows that a gen-

erated row with record π is exactly $\pi^\top A_k z \leq \pi^\top b_k$. A false terminal row therefore has a record λ satisfying $\lambda \geq 0$, $\lambda^\top A_k = 0$, and $\lambda^\top b_k < 0$. ■

Lemma 6 (Lifting the multiplier certificate) *Write the original fixed-rank formulation in the edge variables $\mu \in \mathbb{R}^{E_k}$ as $B_k \mu = b$ and $H_k \mu \leq h$, where the equalities impose agent and object balance and the inequalities impose the edge bounds and rank-prefix comparisons. Each multiplier certificate for the reduced cycle-coordinate system lifts to a Farkas certificate for this original formulation by adding unrestricted multipliers on the balance equalities.*

Proof. Substituting $\mu = \mu^0 + C_k z$ gives $H_k C_k z \leq h - H_k \mu^0$. Let λ be a reduced-system certificate. Then

$$\lambda \geq 0, \quad \lambda^\top H_k C_k = 0, \quad \lambda^\top (h - H_k \mu^0) < 0.$$

By Lemma 2, the columns of C_k span $\ker B_k$. Hence $\lambda^\top H_k$ annihilates $\ker B_k$ and belongs to the row space of B_k . There is an unrestricted vector v such that $\lambda^\top H_k + v^\top B_k = 0$. Since $B_k \mu^0 = b$,

$$\lambda^\top h + v^\top b = \lambda^\top (h - H_k \mu^0) < 0.$$

Thus (λ, v) is a Farkas certificate for the original formulation. Only the inequality multiplier λ is required to be nonnegative. ■

Proof of Theorem 2 Fix a rank k . If the weighted Hall condition fails, Proposition 2 gives $\mathcal{M}_k = \emptyset$, and hence $\mathcal{E}_k = \emptyset$. Suppose that the condition holds. The maximum-flow step returns $\mu^0 \in \mathcal{M}_k$, and the exactness lemma identifies $\mathcal{E}_k$ with the solution set of the cycle-coordinate system. The elimination-and-recovery lemma then gives an exact alternative. A false terminal row yields the stated nonnegative multiplier certificate; if no such row appears, backward recovery yields an allocation in $\mathcal{E}_k$. The lifting lemma establishes the corresponding certificate in the original transportation variables.

It remains to prove termination and rank optimality. At rank $|S|$, define $\bar{\mu}_s(d) = q_s/|D|$ for each $d \in D$ and $s \in S$. Since $\sum_{s \in S} q_s = |D|$ in the normalized equal-claim domain, $\bar{\mu}$ is feasible. Each agent receives the same allotment, so $\bar{\mu}$ satisfies no-envy. The rank scan therefore terminates by rank $|S|$.

Each rejected rank is infeasible for no-envy, while the returned allocation μ^* belongs to $\mathcal{E}_{k^*}$. If $k^* > 1$ and $\rho(\mu^*) < k^*$, then $\rho(\mu^*) \leq k^* - 1$, so $\mu^* \in \mathcal{E}_{k^*-1}$, contradicting the recorded obstruction. If $k^* = 1$, positive claims imply $\rho(\mu^*) = 1$. Hence $\rho(\mu^*) = k^* = k^{NE}$. ■

Proof of Theorem 3 Suppose first that $\mu \in \mathcal{E}_2$. If two agents rank the same object s first, first-rank no-envy in both directions makes their shares of s equal. For each s with $m_s > 0$, denote this common share by $1 - z_s$. Each agent d with $f_d = s$ must receive its remaining amount z_s from g_d . Thus $0 \leq z_s \leq 1$, the recovery formula in the theorem holds, and we may set $z_s = 0$ when $m_s = 0$.

Fix an object $s' \in S$. Agents ranking s' first contribute $m_{s'}(1 - z_{s'})$, while agents ranking an object s first and s' second contribute $c_{ss'}z_s$. Object feasibility is therefore equivalent to

$$m_{s'}(1 - z_{s'}) + \sum_{s \in S} c_{ss'}z_s = q_{s'}.$$

If $c_{ss'} > 0$ and $m_{s'} > 0$, an agent ranking s' first receives $1 - z_{s'}$ from s' , whereas an agent ranking s first and s' second receives z_s from s' . First-rank no-envy gives $z_s + z_{s'} \leq 1$. This proves necessity.

Conversely, suppose that z satisfies the three stated conditions, and construct μ by the recovery formula. The bounds give nonnegativity, and each agent receives one unit. The balance equations give the capacity of each object. The allocation is supported within rank two.

Fix an agent whose first choice is s' . Its own allotment contains $1 - z_{s'}$ units of s' . Another agent receives either the same amount because it also ranks s' first, an amount z_s because it ranks some s first and s' second, or no amount from s' . The local inequalities make each second-choice amount weakly smaller than $1 - z_{s'}$. Thus each first-rank no-envy comparison holds. Comparisons at rank two and each lower-priority threshold are automatic because each agent has received its full claim. Corollary 1 gives $\mu \in \mathcal{E}_2$.

If $|S| = 1$, total balance gives $q_s = |D|$ for the unique object s , and the unique feasible allocation assigns one unit from s to each agent. This allocation satisfies no-envy. ■

Proof of Proposition 4 Let $D = \{d_1, d_2, d_3\}$ and $S = \{s_1, s_2, s_3\}$. Give the objects capacities $(3/2, 1, 1/2)$, and let

$$d_1 : s_1 \succ s_2 \succ s_3,$$

$$d_2 : s_1 \succ s_3 \succ s_2,$$

$$d_3 : s_2 \succ s_1 \succ s_3.$$

Rank one is infeasible because d_1 and d_2 jointly demand two units from s_1 , whose capacity is $3/2$. At rank two, $m_{s_1} = 2$, $m_{s_2} = 1$, and $m_{s_3} = 0$, while $c_{s_1s_2} = c_{s_1s_3} = c_{s_2s_1} = 1$. The balance equation for s_3 gives $z_{s_1} = 1/2$, and the balance equation for s_2 then gives $z_{s_2} = 1/2$. The

balance equation for s_1 holds, and the only binding local inequality is $z_{s_1} + z_{s_2} = 1$. Theorem 3 therefore gives the unique rank-two no-envy allocation

$$\mu^2 = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}, \quad (38)$$

where the columns correspond to s_1, s_2, s_3 .

The allocation μ^2 is not Pareto efficient. For each sufficiently small $\varepsilon > 0$, transfer ε units of s_1 from d_3 to d_1 and the same amount of s_2 from d_1 to d_3 . Agents d_1 and d_3 each replace their second choice by their first choice, while d_2 is unchanged. Since (38) is the unique rank-two no-envy allocation, no rank-two allocation satisfies both no-envy and Pareto efficiency.

Finally, consider

$$\mu^3 = \begin{pmatrix} 3/4 & 1/8 & 1/8 \\ 3/4 & 0 & 1/4 \\ 0 & 7/8 & 1/8 \end{pmatrix}, \quad (39)$$

where the columns correspond to s_1, s_2, s_3 . This allocation is feasible and has social worst rank three.

For d_1 , its cumulative shares at the first two ranks are $3/4$ and $7/8$; the corresponding shares of d_2 are $3/4$ and $3/4$, and those of d_3 are 0 and $7/8$. For d_2 , its cumulative shares are $3/4$ and 1 ; the corresponding shares of d_1 are $3/4$ and $7/8$, and those of d_3 are 0 and $1/8$. For d_3 , its cumulative shares are $7/8$ and $7/8$; the corresponding shares of d_1 are $1/8$ and $7/8$, and those of d_2 are 0 and $3/4$. Corollary 1 therefore implies that μ^3 satisfies no-envy.

The object-precedence relation induced by μ^3 contains only the arcs $s_1 \triangleright_{\mu^3} s_2$, $s_1 \triangleright_{\mu^3} s_3$, and $s_2 \triangleright_{\mu^3} s_3$. It is therefore acyclic. Lemma 3 implies that μ^3 is Pareto efficient. Thus $k^{NE+PE} \leq 3$. The preceding argument rules out a rank-two no-envy and Pareto-efficient allocation, so $k^{NE} = 2 < 3 = k^{NE+PE}$. ■

Proof of Theorem 4. Suppose first that a rank-two no-envy allocation μ exists. Since μ is supported on G_2 , the graph G_2 satisfies Hall's condition. For each $s \in S^+$, first-rank no-envy in both directions implies that all agents with first choice s receive the same share of s . Write this share as $1 - \beta_s$. Each agent d with $a_d = s$ receives its remaining share β_s from b_d . Define $f_s = m_s \beta_s$.

For each $s \in S^+$, object feasibility gives

$$m_s(1 - \beta_s) + \sum_{s' \in S^+} e_{s's} \beta_{s'} = 1, \quad \text{or equivalently} \quad f_s = m_s - 1 + \sum_{s' \in S^+} P_{s's} f_{s'}. \quad (40)$$

For each $s \in S^0$, object feasibility gives

$$\sum_{s' \in S^+} P_{s's} f_{s'} = 1. \quad (41)$$

If $e_{ss'} > 0$ and $m_{s'} > 0$, an agent ranking s' first receives $1 - \beta_{s'}$ from s' , whereas an agent ranking s first and s' second receives β_s from s' . First-rank no-envy therefore gives $\beta_s + \beta_{s'} \leq 1$.

Fix a closed class C . Each agent whose first choice lies in C also has its second choice in C . Hall's condition applied to these agents gives $\sum_{s \in C} m_s \leq |C|$. Since $m_s \geq 1$ for each $s \in C$, we have $m_s = 1$ throughout C . Each vertex then has exactly one outgoing arc, and strong connectivity makes C a directed cycle.

Each row of P sums to one, while P_T is substochastic. From each state in T , a directed path reaches either a closed class in R or an object in S^0 . Otherwise, the set reachable from that state would contain a closed strongly connected component inside T , contrary to the definition of R . If T is nonempty, finiteness and the existence of an exit path from each state imply that there are an integer $L \geq 1$ and a number $\eta > 0$ such that each state exits T within L steps with probability at least η . Hence $\|P_T^L\|_\infty \leq 1 - \eta < 1$. Thus P_T is transient, its spectral radius is below one, and $I - P_T^\top$ is nonsingular. Since no arc leaves R , restricting (40) to T gives

$$(I - P_T^\top) f_T = m_T - \mathbf{1}.$$

Hence each rank-two no-envy allocation induces the same restriction f_T , namely the T -component of the canonical vector. The canonical representative sets $f_s = 0$ for each $s \in R$.

Summing (40) over a closed class cancels its internal displaced flow. The remaining external inflow is nonnegative and must equal zero, which is condition (25). Equation (41) gives condition (26). The bounds $0 \leq \beta_s \leq 1$ give condition (27), and the first-rank no-envy inequalities give condition (28). This proves necessity.

Conversely, suppose that Hall's condition holds and that the canonical vector satisfies conditions (25)–(28). The allocation $\hat{\mu}$ has unit row sums and is nonnegative because $0 \leq f_s \leq m_s$ on T , while $\beta_s = 0$ on R . Equation (24) gives object feasibility on T . Each $s \in R$ has $m_s = 1$ and $\beta_s = 0$, so its first-choice agent contributes one unit; condition (25) excludes external displaced inflow. Condition (26) gives object feasibility on S^0 . Thus $\hat{\mu}$ is feasible and rank-two supported.

Only first-rank no-envy can be nontrivial. Agents with the same first choice receive the same first-choice share. An outsider who receives s' as a second choice receives β_s along an arc $s \rightarrow s'$. If $s, s' \in T$, condition (28) gives $\beta_s \leq 1 - \beta_{s'}$. If $s' \in R$, condition (25) and nonnegativity imply $P_{ss'} f_s = 0$, so no positive outsider share enters s' . No arc leaves a closed class. If $s' \in S^0$, no agent ranks s' first, so s' generates no first-rank comparison. Hence each

first-rank no-envy inequality holds. Rank-two comparisons are automatic because each agent receives one unit. Therefore, $\hat{\mu}$ is no-envy. ■

Proof of Theorem 5 If $k^{NE} = 1$, a rank-one feasible allocation gives each agent its first choice and is ordinally efficient. Hence $k^{NE+OE} = 1$. Suppose that $k^{NE} = 2$, and fix a rank-two no-envy allocation μ .

For each $s \in S$, let β_s be the common displacement from the first choice among agents ranking s first, and set $\beta_s = 0$ when $m_s = 0$. Object feasibility and first-rank no-envy give

$$m_s \beta_s - \sum_{s' \in S} e_{s's} \beta_{s'} = m_s - 1 \quad \text{for each } s \in S, \quad (42)$$

$$\beta_s + \beta_{s'} \leq 1 \quad \text{if } e_{ss'} > 0 \text{ and } m_{s'} > 0. \quad (43)$$

Construct the active graph on S by including an arc $s \rightarrow s'$ whenever $e_{ss'} > 0$ and $\beta_s > 0$. This graph is the nontrivial part of the object-precedence relation induced by μ . If it is acyclic, Lemma 3 implies that μ is ordinally efficient.

Suppose that the active graph contains a directed cycle. Let U be the set of objects from which that cycle is reachable along active arcs. Each $s \in U$ has an active arc to another member of U , so

$$\sum_{s \in U} \sum_{s' \in U} e_{ss'} \geq |U|.$$

Moreover, no positive displaced flow enters U from outside: if $s \notin U$, $s' \in U$, and $e_{ss'} \beta_s > 0$, then the active arc $s \rightarrow s'$ would put s in U .

Summing (42) over $s \in U$ gives

$$\sum_{s \in U} \sum_{s' \notin U} e_{ss'} \beta_s = \sum_{s \in U} (m_s - 1). \quad (44)$$

Let $B(U) = \sum_{s \in U} \sum_{s' \notin U} e_{ss'}$. The internal-arc inequality gives $B(U) \leq \sum_{s \in U} (m_s - 1)$, while (44) and $\beta_s \leq 1$ give the reverse inequality. Equality holds throughout.

If an arc left U , equality would require its tail s to satisfy $\beta_s = 1$. Choose the first active arc $s \rightarrow s'$ on a path from s to the fixed cycle. Since $s' \in U$, it has an outgoing active arc and therefore $\beta_{s'} > 0$. The inequality (43) for $s \rightarrow s'$ would give $1 + \beta_{s'} \leq 1$, a contradiction. Hence no arc leaves U . It follows that $B(U) = 0$ and $m_s = 1$ for each $s \in U$.

Set $\tilde{\beta}_s = 0$ for each $s \in U$ and $\tilde{\beta}_s = \beta_s$ otherwise, and construct the associated rank-two allocation. Since no arc leaves U , no positive displaced flow enters U , and $m_s = 1$ on U , object feasibility is preserved. Each displacement weakly decreases, so all first-rank no-envy inequalities remain valid. The resulting allocation is feasible, rank-two supported, no-envy, and has

removed each active cycle contained in U . Repeating the operation finitely many times yields an allocation with an acyclic object-precedence relation. It is ordinally efficient by Lemma 3. Therefore $k^{NE+OE} \leq 2 = k^{NE}$, and the reverse inequality follows from the definitions. ■

Proof of Theorem 6 Since claims are unitary, feasible truncation leaves each competing allotment unchanged, so no-envy coincides with stochastic-dominance envy-freeness. Probabilistic Serial therefore provides a no-envy and ordinally efficient allocation for each strict unit-capacity instance (Bogomolnaia and Moulin, 2001); in particular, $k^{NE+OE} \leq n$.

We first establish the upper bounds. If $k^C = 1$, a rank-one feasible allocation assigns each agent its first-choice object. Unit capacities then make the first-choice map bijective, so the allocation is no-envy and ordinally efficient. Hence $k^{NE} = k^{NE+OE} = 1$. If $k^C \geq 2$, then $k^{NE+OE} \leq n$, and therefore

$$k^{NE} - k^C \leq k^{NE+OE} - k^C \leq n - 2.$$

This proves the upper bounds for the no-envy cost and the combined cost. For the conditional-efficiency cost, if $k^{NE} \leq 2$, Theorem 5 gives $k^{NE+OE} = k^{NE}$. If $k^{NE} \geq 3$, then $k^{NE+OE} \leq n$, so $k^{NE+OE} - k^{NE} \leq n - 3$. This also shows that the conditional-efficiency maximum is zero for $n \leq 3$.

We next prove sharpness of the no-envy and combined bounds. For $n \leq 2$, an instance with distinct first choices attains both zero bounds. Fix $n \geq 3$. Agents d_1 and d_2 rank the objects according to

$$s_1 \succ s_2 \succ \cdots \succ s_n.$$

For each $j \in \{3, \dots, n\}$, agent d_j ranks s_1 first, s_j second, and the remaining objects thereafter in increasing-index order.

The assignment that sends d_1 to s_2 , d_2 to s_1 , and d_j to s_j for each $j \in \{3, \dots, n\}$ is rank-two feasible. Rank one is infeasible because all agents rank s_1 first. Hence $k^C = 2$.

Suppose that a no-envy allocation μ has social worst rank at most $n - 1$. Since all agents rank s_1 first, first-rank no-envy and the capacity of s_1 imply

$$\mu_{s_1}(d) = \frac{1}{n} \quad \text{for each } d \in D.$$

Each agent in $D \setminus \{d_n\}$ ranks s_n last. The support restriction therefore gives $\mu_{s_n}(d) = 0$ for each $d \in D \setminus \{d_n\}$. Object feasibility then requires $\mu_{s_n}(d_n) = 1$. This contradicts the row sum of d_n , because $\mu_{s_1}(d_n) = 1/n > 0$. Thus no no-envy allocation has social worst rank below n . The upper bound $k^{NE+OE} \leq n$ established at the beginning of the proof and the preceding impossibility imply $k^{NE} = k^{NE+OE} = n$.

It remains to prove sharpness of the conditional-efficiency bound. For $n \leq 3$, the zero upper bound is attained by an instance with distinct first choices.

For $n = 4$, consider the profile in (30). Rank two is infeasible because the agents d_1, d_2, d_3 have only s_1 and s_2 among their first two choices. The displayed allocation $\mu^\dagger$ is rank-three supported and no-envy. Each agent receives $1/4$ from the common first choice s_1 ; the agents d_1, d_2, d_3 each receive $1/2$ from $\{s_1, s_2\}$; and d_4 receives $3/4$ from $\{s_1, s_4\}$, weakly more than each other agent receives from that prefix.

The allocation is the unique rank-three no-envy allocation. First-rank no-envy and the capacity of s_1 give $\mu_{s_1}(d) = 1/4$ for each agent d . The agents d_1, d_2, d_3 must receive a common share β from s_2 . Rank-three support permits only d_1 and d_2 to receive s_3 , and mutual no-envy between these two identical agents makes those shares equal. The capacity of s_3 and their row sums give $2(3/4 - \beta) = 1$, so $\beta = 1/4$. All remaining entries follow from the row and column sums. Finally, d_3 receives a positive share of s_4 and prefers s_2 , while d_4 receives a positive share of s_2 and prefers s_4 . Exchanging equal positive amounts of s_2 and s_4 strictly improves both agents. Thus no rank-three no-envy allocation is ordinally efficient. The preceding arguments and the upper bound $k^{NE+OE} \leq 4$ therefore give $k^{NE} = 3$ and $k^{NE+OE} = 4$.

It remains to construct a tight instance for each $n \geq 5$. Let $D = \{d_1, \dots, d_n\}$, let $S = \{s_1, \dots, s_n\}$, and set $E = \{s_5, \dots, s_n\}$. Define the preferences of the first four agents by

$$\begin{aligned} d_1, d_2 : s_1 \succ s_2 \succ s_3 \succ s_4 \succ s_5 \succ \dots \succ s_n, \\ d_3 : s_1 \succ s_2 \succ s_4 \succ s_5 \succ \dots \succ s_n \succ s_3, \\ d_4 : s_1 \succ s_4 \succ s_2 \succ s_5 \succ \dots \succ s_n \succ s_3. \end{aligned}$$

For each $j \in \{5, \dots, n\}$, agent d_j ranks s_j first, ranks the other members of E next in increasing-index order, and then ranks $s_1 \succ s_2 \succ s_4 \succ s_3$.

Define μ^0 as follows. Agents d_1 and d_2 each receive $1/2$ from s_3 and $1/4$ from each of s_1 and s_2 . Agents d_3 and d_4 each receive $1/2$ from s_4 and $1/4$ from each of s_1 and s_2 . Each agent d_j with $j \geq 5$ receives one unit from s_j . This allocation is feasible and rank-three supported. For d_1, d_2, d_3 , the only nontrivial prefixes contain s_1 and then $\{s_1, s_2\}$; the corresponding own masses are $1/4$ and $1/2$, no smaller than the competing masses. For d_4 , the corresponding prefixes are $\{s_1\}$ and $\{s_1, s_4\}$, with own masses $1/4$ and $3/4$. Each remaining agent receives its entire unit from its first choice. Thus μ^0 is no-envy. Rank two is infeasible because $\{d_1, d_2, d_3\}$ has neighborhood $\{s_1, s_2\}$. Hence $k^{NE} = 3$.

Fix a no-envy allocation μ with social worst rank at most $n - 1$. Only d_1 and d_2 can receive s_3 , because each other agent ranks it last. Since d_1 and d_2 have identical strict preferences, no-envy in both directions makes all their rank-prefix totals, and hence all their object shares,

equal. The capacity of s_3 therefore gives

$$\mu_{s_3}(d_1) = \mu_{s_3}(d_2) = \frac{1}{2}. \quad (45)$$

First-rank no-envy among the first four agents gives a common share α of s_1 . No-envy at the prefix $\{s_1, s_2\}$ gives a common share β of s_2 for d_1, d_2, d_3 . The identical preferences of d_1 and d_2 also give a common share $\delta = \mu_{s_4}(d_1) = \mu_{s_4}(d_2)$. Set $x = \mu_{s_4}(d_3)$. No-envy for d_3 at its top-three prefix gives

$$x \geq \delta, \quad \mu_{s_2}(d_4) + \mu_{s_4}(d_4) \leq \beta + x. \quad (46)$$

For each $j \in \{5, \dots, n\}$, the first $n - 2$ objects of d_j form $E \cup \{s_1, s_2\}$. Both d_j and d_3 receive zero from their last-ranked object s_3 under the assumed support bound. Their masses in this prefix are therefore $1 - \mu_{s_4}(d_j)$ and $1 - x$, respectively. No-envy for d_j gives

$$\mu_{s_4}(d_j) \leq x. \quad (47)$$

Using (46) and (47) in the capacity equation for s_4 yields

$$1 = 2\delta + x + \mu_{s_4}(d_4) + \sum_{j=5}^n \mu_{s_4}(d_j) \leq \beta + nx.$$

Equation (45) and the row sum of d_1 give $\alpha + \beta \leq 1/2$. Hence $x \geq 1/(2n) > 0$.

Suppose, toward a contradiction, that μ is ordinally efficient. If $\mu_{s_2}(d_4) > 0$, then d_3 and d_4 generate the trading cycle $s_2 \rightarrow s_4 \rightarrow s_2$. Thus $\mu_{s_2}(d_4) = 0$. The agents d_3 and d_4 have the same top-three set $\{s_1, s_2, s_4\}$. No-envy in both directions equates their cumulative shares on this set and gives

$$\mu_{s_4}(d_4) = \beta + x. \quad (48)$$

Agent d_4 must receive a positive share from some object $s \in E$. Otherwise, its row sum, the support bound, and (48) give $\alpha + \beta + x = 1$. The row sum of d_1 gives $x \geq 1/2$, while the capacity of s_1 gives $\alpha \leq 1/4$. Hence

$$x + \mu_{s_4}(d_4) = x + 1 - \alpha > 1,$$

contradicting the capacity of s_4 .

Fix $s \in E$ such that $\mu_s(d_4) > 0$. The objects in E have total capacity $n - 4$. Since d_4 receives a positive amount from this set, some agent d_j with $j \geq 5$ must receive a positive amount from s_1, s_2 , or s_4 ; the support bound excludes s_3 . If such an agent receives s_4 , it and d_4 generate the cycle $s \rightarrow s_4 \rightarrow s$. If no such agent receives s_4 but one receives s_2 , that agent, d_3 , and d_4 generate $s \rightarrow s_2 \rightarrow s_4 \rightarrow s$. In the remaining case, some d_j receives s_1 , and no agent

with index at least five receives s_2 or s_4 . The capacity of s_2 and $\mu_{s_2}(d_4) = 0$ then imply $3\beta = 1$. Agents d_j, d_1, d_3, d_4 generate the cycle $s \rightarrow s_1 \rightarrow s_2 \rightarrow s_4 \rightarrow s$. Each case contradicts Lemma 3.

No no-envy and ordinally efficient allocation can therefore have social worst rank below n . Together with the upper bound $k^{NE+OE} \leq n$, this proves $k^{NE+OE} = n$. The constructed family attains the upper bound $n - 3$. ■

Proof of Theorem 7. Fix $X \in \{OE, NE, NE + OE\}$. If $k^X = 1$, a rank-one feasible allocation assigns each agent its first choice. Unit capacities then make the first-choice map bijective. Under PS, each agent consumes its distinct first choice until completion, so $k^{PS} = 1$. Thus $k^{PS} - k^X = 0$ and $k^{PS}/k^X = 1$. If $k^X \geq 2$, then $k^{PS} \leq n$, and therefore

$$k^{PS} - k^X \leq n - 2, \quad \frac{k^{PS}}{k^X} \leq \frac{n}{2}.$$

It remains to construct one family attaining all six bounds.

Fix $n \geq 3$, let $D = \{d_1, \dots, d_n\}$, and let $S = \{s_1, \dots, s_n\}$. Define

$$\begin{aligned} d_1 &: s_1 \succ s_2 \succ \dots \succ s_n, \\ d_r &: s_r \succ s_{r+1} \succ \dots \succ s_n \succ s_1 \succ \dots \succ s_{r-1} && \text{for } r \in \{2, \dots, n-1\}, \\ d_n &: s_1 \succ s_n \succ s_2 \succ \dots \succ s_{n-1}. \end{aligned}$$

Define μ° by

$$\begin{aligned} \mu_{s_1}^\circ(d_1) &= \mu_{s_2}^\circ(d_1) = \frac{1}{2}, \\ \mu_{s_r}^\circ(d_r) &= \mu_{s_{r+1}}^\circ(d_r) = \frac{1}{2} && \text{for } r \in \{2, \dots, n-1\}, \\ \mu_{s_1}^\circ(d_n) &= \mu_{s_n}^\circ(d_n) = \frac{1}{2}, \end{aligned}$$

with all remaining entries equal to zero. Each object supplies exactly two agents with one half unit each, so μ° is feasible. Each agent receives one half of its first choice and one half of its second choice. At the first-rank threshold, no other allotment contains more than one half of the evaluating agent's first choice; at the second-rank threshold, the evaluating agent has received its full unit. Hence μ° is no-envy.

The nontrivial object-precedence arcs are $s_1 \rightarrow s_2$, $s_r \rightarrow s_{r+1}$ for $r \in \{2, \dots, n-1\}$, and $s_1 \rightarrow s_n$. Each arc points from a lower-indexed object to a higher-indexed object. The relation is acyclic, so μ° is ordinally efficient. Rank one is infeasible because d_1 and d_n share first choice s_1 . Consequently,

$$k^{OE} = k^{NE} = k^{NE+OE} = 2. \tag{49}$$

We now trace PS. For each $r \in \{1, \dots, n-1\}$, set $t_r = 1 - 1/(2r)$. During $[0, t_1]$, agents d_1 and d_n consume s_1 , while each agent d_r with $r \in \{2, \dots, n-1\}$ consumes s_r . Thus s_1 is exhausted at $t_1 = 1/2$.

For each $r \in \{2, \dots, n-1\}$, agents $d_1, \dots, d_r$ consume s_r throughout $[t_{r-1}, t_r]$, and s_r is exhausted at t_r . To prove this claim by induction, suppose that s_{r-1} is exhausted at t_{r-1} . Agents $d_1, \dots, d_{r-1}$ then move to s_r , while d_r has consumed s_r since time zero. The residual supply of s_r is $1 - t_{r-1} = 1/[2(r-1)]$. The aggregate consumption rate is r , and $t_r - t_{r-1} = 1/[2r(r-1)]$. Hence $r(t_r - t_{r-1}) = 1 - t_{r-1}$, proving the claim.

In particular, s_{n-1} is exhausted at $t_{n-1} = 1 - 1/[2(n-1)]$. Agent d_1 then moves to s_n , its last choice. Agent d_n has consumed s_n since time t_1 . The remaining supply of s_n at t_{n-1} is

$$1 - (t_{n-1} - t_1) = \frac{n}{2(n-1)}.$$

All n agents consume s_n during the remaining interval, whose length is $1 - t_{n-1} = 1/[2(n-1)]$. The residual supply is exactly n times this length, so s_n is exhausted when all agents complete. In particular, $\mu_{s_n}^{PS}(d_1) = 1/[2(n-1)] > 0$, and therefore $k^{PS} = n$. Together with (49), this proves equality in each additive and multiplicative bound. ■

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